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I. Introduction

A charged particle beam line is simply a set of magnets placed in a predetermined sequence and used to transmit particles. The trajectories of the particles begin at some point and traverse each magnet in turn until they arrive at a final point. The beam, or envelope, of all such trajectories passed by the beam line, will have characteristics at this final point which are determined by the nature of the various magnets. The object of any design is to determine the magnet characteristics necessary to achieve certain desired beam properties at the end of the system.

The computations are usually done by expressing the final coordinates of a trajectory as a Taylor series expansion of the initial coordinates. Since a set of six coordinates is used the coefficients of the expansion are matrices. For systems with large initial phase space it is necessary to consider higher order terms.

Such large initial phase space configurations occur more often in lower energy beams. The ability to calculate analytically the matrix coefficients would be a great aid in designing such beams. Such beams would be of use in cancer therapy, as high resolution electron microscopes, and in nuclear and particle physics.

The first[1] and second[2] order terms in this solution have been worked out and are employed in creating beam line designs. Higher order matrix coefficients are sufficiently complicated so as to require algebraic manipulation by computer. We elaborate below in greater detail on how to obtain such solutions.

II. The equations of motion

The motion of a charged particle in a static magnetic field is given by setting the time derivative of the momentum equal to the Lorentz Force.

$$\dot{p} = q \, \underline{v} \times \underline{B} \tag{1}$$

It is convenient to solve this equation in a special coordinate system. For this purpose we define a reference trajectory of a given momentum p_o which begins at the origin and passes through each magnet in turn. In traversing a given magnet it experiences uniform magnetic field. In this sense, both magnets and beam line are somewhat idealized.

To determine the coordinates of a given point we construct a plane containing this point through which the central trajectory passes perpendicularly. The longitudinal coordinate t is the length of the central trajectory up to the point of intersection with the plane. The transverse coordinates x and y are the two coordinates in the constructed plane with the reference trajectory passing through the origin. To describe the behavior of a trajectory at a given point we employ a vector of six coordinates.

$$\underline{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ \ell \\ \delta \end{pmatrix} \tag{2}$$

The newly appearing quantities are x' and y' which are dx/dt and dy/dt respectively, ℓ which is the difference in longitudinal position from a particle on the central trajectory for equal times of flight, and δ the fractional derivation of the momentum from that of the central trajectory.

If we then expand the equations of motion in powers of these coordinates we arrive at

$$x'' + k_x^2 x = \text{second and higher order terms} \qquad (3)$$

$$y'' + k_y^2 y = \text{second and higher order terms.} \qquad (4)$$

Hence forth we will illustrate the problems by referring only to equation (3). Equation (4) is handled similarly. The second and higher order terms may be represented symbolically as

$$\text{second and higher order terms} = Exx + Fxxx + \ldots \qquad (5)$$

where E is a matrix with three indices and Exx means

$$\sum_{jk} E_{ijk}\, X_j\, X_k \,,$$

the index i indicating the coordinate to which the equation applies. The matrix F has four indices and is defined similarly.

III. Solution of the Equations

A. First Order

The first order equations of motion are obtained by taking only those terms linear in the six coordinates. The solutions to this equation are also linear and the final set of coordinates of a trajectory can be given as a transfer matrix times the initial set.

$$X_i(1) = \sum_j R_{ij}\, X_j\,(0) \qquad (6)$$

The matrix R now represents the first order properties of the beam line. Beam lines are generally designed so that certain elements of the R matrix will have certain specified values.

Two specific solutions are of interest and will be referred to later. They are the sine-like trajectory S_x with initial values $S_x(0) = 0$, $S_x'(0) = 1$, and the cosine-like trajectory with initial values $C_x(0) = 1$, $C_x'(0) = 0$.

The R matrix for an entire system can be obtained by multiplying the matrices for the individual elements. Individual R matrices for different types of magnets have been evaluated analytically.[2]

B. Second Order Terms

To determine the second order terms, namely the quadratic dependences of the final coordinates on the initial ones, we now substitute the first order solutions for the coordinates into the second order terms on the right side of equations (3) and (4) so that

$$\sum_{jk} E_{ijk} X_j(t)\, X_k(t) = \sum_{jk\ell m} E_{ijk} R_{j\ell}(t)\, R_{km}(t)\, X_\ell(0)\, X_m(0) \qquad (7$$

If we now define a Green's function by

$$G_x(t,\zeta) = S_x(t)\, C_x(\zeta) - S_x(\zeta)\, C_x(t) \qquad (8$$

we can obtain second order matrix elements in the form

$$T_{i\ell m}(t) = \int_0^t G_x(t,\zeta) \sum_{jk} E_{ijk} R_{j\ell}(\zeta) R_{km}(\zeta)\, d\zeta \qquad (9$$

The coordinates for a trajectory at the final point in a system may now be expressed to second order in terms of the initial coordinates as

$$X_i(1) = \sum_j R_{ij} X_j(0) + \sum_{jk} T_{ijk} X_j(0) X_x(0) \tag{10}$$

Second order transfer matrix elements for various types of magnets have been worked out analytically.[2] Once again, the first and second order matrices for an entire system can be obtained by multiplying together the matrices for individual elements.

IV. Future Extensions to Higher Orders

Once solutions for a given order have been obtained, the next higher order may be obtained by the iterative technique described above. To obtain the third order matrices, one would substitute the second order solutions into the right hand sides of equations (3) and (4). The terms contributing to the third order matrices would be

$$K_{imnp} = \sum_{jk} E_{ijk} \left[R_{jm}(t)\; T_{knp}(t) + T_{jmn}(t)\; R_{kp}(t)\right] + \sum_{jk\ell} F_{ijk\ell}\; R_{jm}(t)\; R_{kn}(t)\; R_{\ell p}(t) \tag{11}$$

and the third order matrices themselves would be given as

$$U_{imnp} = \int_0^t G_x(t,\zeta)\; K_{imnp}(\zeta)\; d\zeta \tag{12}$$

Extension to fourth order would be done by a similar iteration using the third order solutions. As before, the matrices for an entire system of a given order could be obtained by multiplying together matrices for individual elements.

The third order coefficients in the equations of motion in an arbitrary bending magnet have been derived.[3] The corresponding fourth order terms have not been derived. Similarly the transfer matrices for third and fourth order have not been evaluated. Since six coordinates are involved the number of different matrix elements of third or fourth order is great, and the complexity of each term increases substantially as the order increases.

Therefore, we must conclude that the only practical way to evaluate third and fourth order matrix elements is by using algebraic manipulation techniques on a computer. We would first expand the equations of motion to fourth order and collect terms. Then we could isolate given terms, multiply by the Green's function, and perform the integrals by using substitution techniques. The final expressions could be incorporated into beam optics computer programs.

The number of fundamentally different types of magnet is small, so the integrals need be evaluated only a small number of times. However, once done, the resulting advances in the state of the art of charged particle beam design would be great.

References

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